V 4-manifolds

A. Kirby diagrams

given a 4-manifold X
it has a handle decomposition with
one 0-handle
one 4-handle
some number of 1, 2, and 3-handles

0-handle

$h^0 = D^0 \times D^4$
$\partial_- h^0 = \emptyset$

note other handles attached to
$$\partial_+ h^0 = S^3$$

we can assume attaching spheres for all other handles (except 4-handle) miss a point $p \in S^3$

so we draw them in $\mathbb{R}^3 = S^3 - \{p\}$

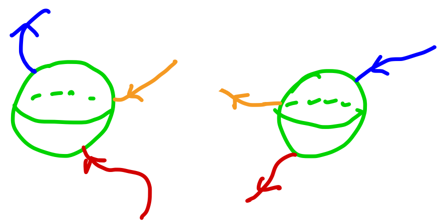
1-handles

$h^1 = D^1 \times D^3$
$\partial_- h^1 = S^0 \times D^3$
framing (if X orientable) in $\pi_0(SO(3)) = \{1\}$

so for each 1-handle you see the following in $\mathbb{R}^3$

and you interpret this as identifying the balls by reflection

e.g.



2-handles

$h^2 = D^2 \times D^2$

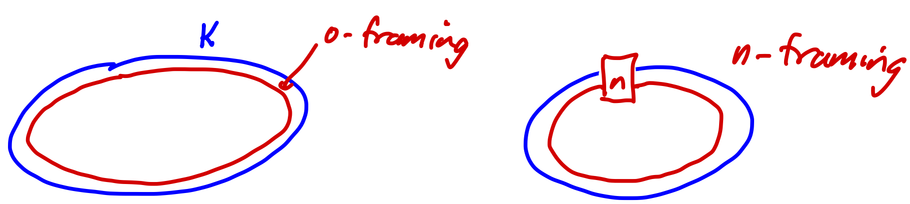
$\partial_- h^2 = S^1 \times D^2$

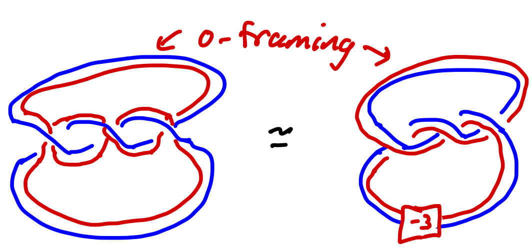
framings $\longleftrightarrow$ $\pi_1(SO(2)) \cong \mathbb{Z}$

recall, to think of framings as $\mathbb{Z}$ we need to fix a framing

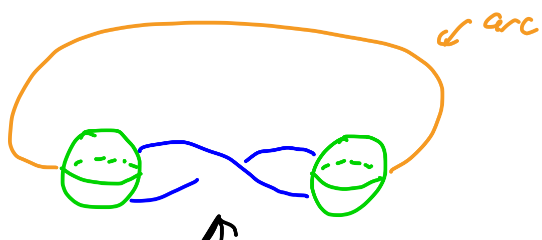
if $K \subset \mathbb{R}^3$, then $K = \partial \Sigma$, Σ a Seifert surface

so we use this as the 0-framing and all others differ by "twists"



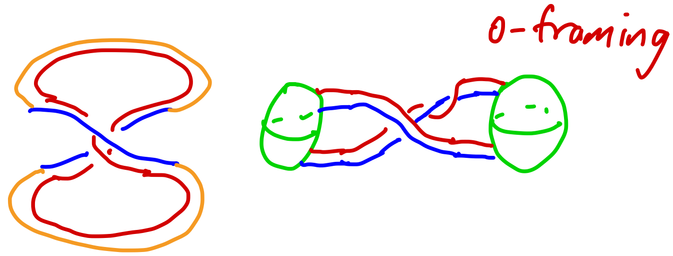


if K goes over 1-handle then we need to choose an arc to determine 0-framing



Keep K in region

"close" K with arc to see 0-framing



if X is a closed 4-manifold then after attaching the 0, 1, and 2-handles we migh need to attach some 3-handles and a 4-handle

exercise: 3-handles $\cup$ 4-handle $\cong$ 0-handle $\cup$ 1-handles

so $\cong \natural_k S^1 \times D^3$

boundary sum

3-handles

so ∂(3 & 4 handles) $\cong \#_k S^1 \times S^2$

Th$^{\underline{m}}$ (Laudenbach–Poénaru):

$\boxed{\text{any diffeomorphism of } \#_k S^1 \times S^2 \text{ extends over } \natural_k S^1 \times D^3}$

from this we get

Th$^{\underline{m}}$ 1:

suppose X, X' are closed 4-manifolds and

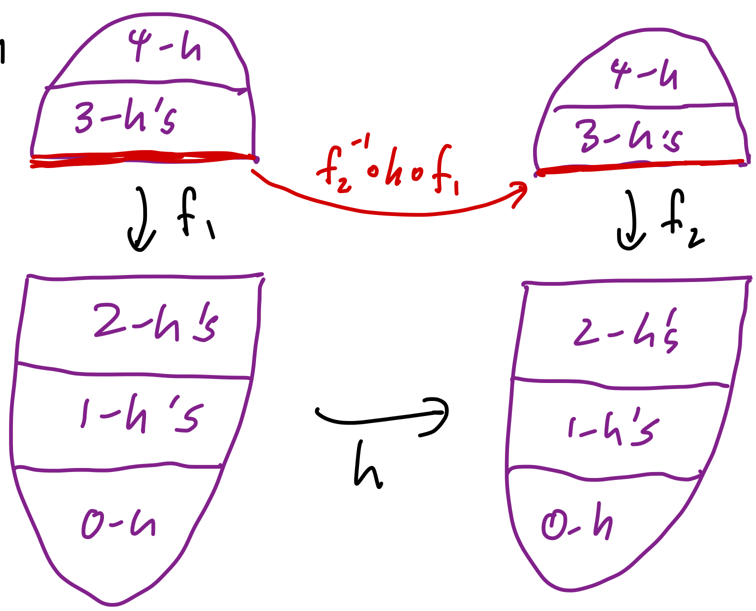
X_i = union of handles of index $\leq i$

(similarly for X_i')

if $X_2 \cong X_2'$ then $X \cong X'$

Proof: let $h: X_2 \to X_2'$ be diffeomorphism

then



th$^{\underline{m}}$ above says $\exists\, F: (3 \text{ and } 4\text{-h's}) \to (3 \text{ and } 4\text{-h's})$

that extends $f_2^{-1} \circ h \circ f_1$

can use h and F to get diffeo $X \to X'$ ▨

So if we are interested in closed 4-manifolds then we can ignor the 3 and 4-handles!

but if you want to consider non-closed 4-manifolds you need to see how the 3-handles are attached

3-handles

$$h^3 = D^3 \times D^1$$
$$\partial_- h^3 = S^2 \times D^1$$

framings $\leftrightarrow \pi_2(SO(1)) = \{1\}$

so you just need to find an embedded $S^2 \subset \partial X_2$

then unique way to attach 3-handle

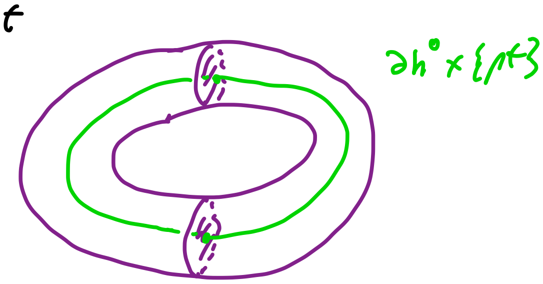
examples:

1) $S^2 \times D^2$ we do this just like we did for 3-manifolds

so start with $S^2 =$

now $h^0 \times D^2$ can be thought of as a 4D 0-handle

note: in its boundary we see $(\partial h^0) \times D^2$ is just

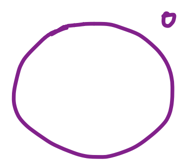


so this is the attaching sphere for $h^2 \times D^2$

i.e. $(\partial h^2) \times D^2$ is glued to $(\partial h^0) \times D^2$

and since we have a global product we identify the product str on $(\partial h^2) \times D^2$ with the one on $(\partial h^0) \times D^2$

i.e. we use the 0-framing

so $S^2 \times D^2$ is 

exercise: in this picture "see" that

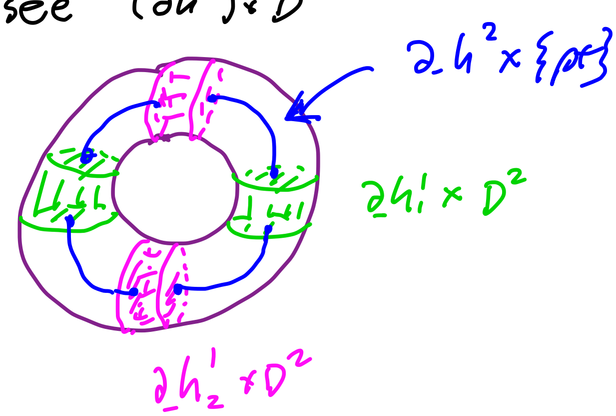
$$\partial(S^2 \times D^2) = S^2 \times S^1$$

that is "see" an S^1's of S^2's

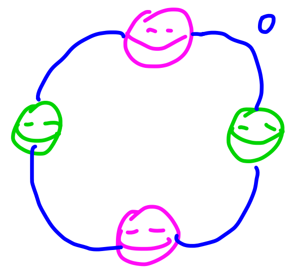
2) $T^2 \times D^2$ again arguing as we did for $T^2 \times I$ we see

$$T^2 \times D^2 = \underbrace{(h^0 \times D^2)}_{0\text{-}h} \cup \underbrace{(h_1^1 \times D^2) \cup (h_2^1 \times D^2)}_{1\text{-handles}} \cup \underbrace{(h^2 \times D^2)}_{2\text{-}h}$$

in $\partial(0\text{-}h)$ we see $(\partial h^0) \times D^2$

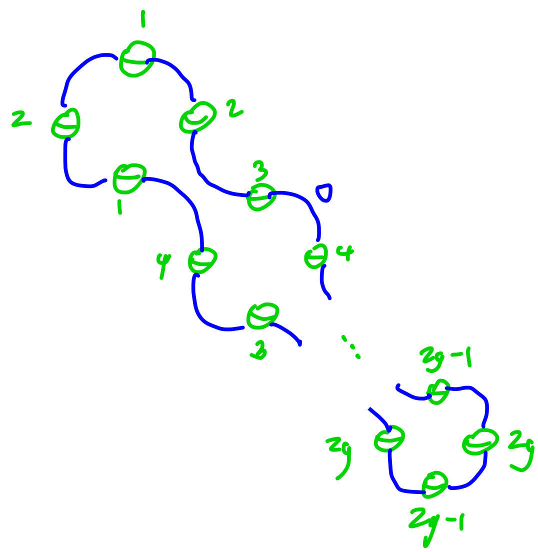


so $T^2 \times D^2$ is



exercise: "see" $\partial(T^2 \times D^2) = T^2 \times S^1 = T^3$

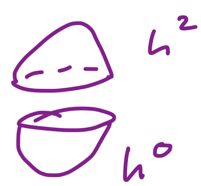
3) **exercise:** $\Sigma_g \times D^2$ surface of genus g



4) D^2-bundles over S^2

let E be a D^2-bundle over S^2

note: $S^2 =$



$E|_{h^i} = h^i \times D^2$ since any bundle over a contractible space is trivial

now we need to glue $\partial h^2 \times D^2$ to $\partial h^0 \times D^2$ by a diffeomorphism that respects the fibration

$$\begin{array}{ccc} \partial h^2 \times D^2 & \xrightarrow{F} & \partial h^0 \times D^2 \\ \wr\| & & \wr\| \\ S^1 \times D^2 & & S^1 \times D^2 \\ (\phi, (r,\theta)) & \longmapsto & (\phi, f(\phi)(r,\theta)) \end{array}$$

here $f: S^1 \to \mathrm{Diff}(D^2)$

recall if we homotop f then the resulting bundle is unchanged

so we can think of f as an element of

$$\pi_1(\mathrm{Diff}(D^2)) \cong \pi_1(SO(2)) \cong \mathbb{Z}$$

$\simeq SO(2)$

so can take f to be

$$f_\phi(r,\theta) = (r, \theta + n\phi)$$

$\therefore$ f_ϕ is determined by n and E is determined by f_ϕ $\quad E = (h^2 \times D^2) \cup_F (h^0 \times D^2)$

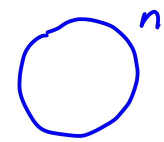
so for each n we get a D^2-bundle E_n over S^2 and any D^2-bundle is equivalent to E_n for some n

exercise: show E_n is orientation preserving diffeomorphic to $E_m \Leftrightarrow n = m$

Hint: wait till later and consider the "intersection form"

exercise:

E_n is given by the Kirby diagram

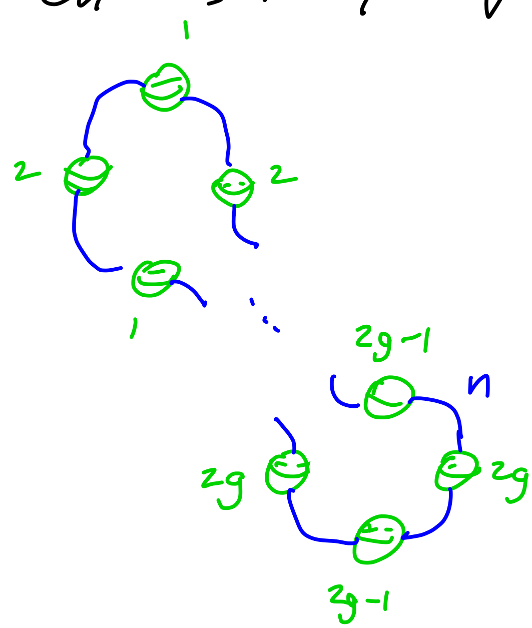


Hint: consider what framing means!

5) exercise:

1) show D^2-bundles over a surface Σ_g, of genus g are also determined by an integer (still denote E_n)

2) show E_n has Kirby diagram



lemma 2:

if Σ is a surface embedded in a

4-manifold X, then there exist a neighborhood of Σ in X diffeomorphic to a D^2-bundle over Σ

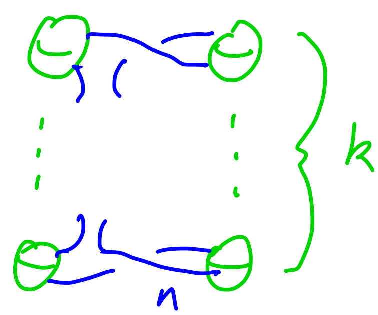
Proof: exercise

so we understand Kirby diagrams of nbhds of any surface in a 4-manifold!

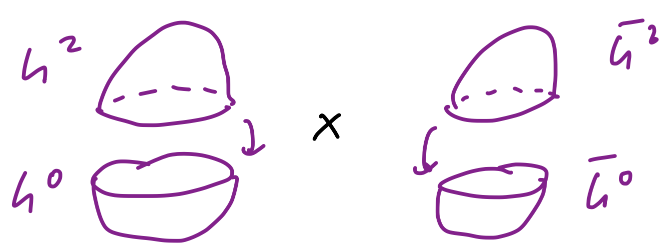
6) exercise:

1) Show there are an integers worth of D^2-bundles over a non-orientable surface N_k (= # of k projective planes)

2) Show a Kirby diagram for them is

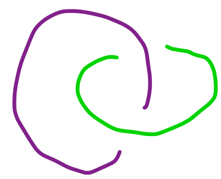


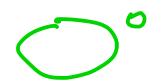
7) $S^2 \times S^2$

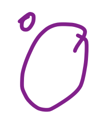


so $S^2\times S^2 = \overbrace{(h^0\times \bar{h}^0)}^{0-h} \cup \overbrace{(h^0\times \bar{h}^2)\cup(h^2\times \bar{h}^0)}^{2\text{ handles}} \cup \overbrace{(h^2\times \bar{h}^2)}^{4-h}$

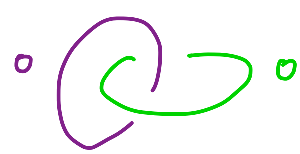
exercise: show $(\underline{\partial h^0\times\{pt\}}) \cup (\underline{\{pt\}\times \partial \bar{h}^0})$ in
$\mathbb{R}^3 \subset S^3 = \partial(h^0\times \bar{h}^0)$ is

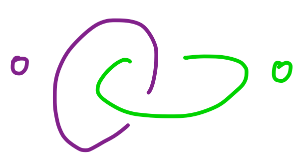


now $(h^0\times\bar{h}^0)\cup(h^0\times\bar{h}^2)$ is $S^2\times D^2$ so 

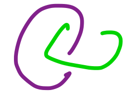
and $(h^0\times\bar{h}^0)\cup(h^2\times\bar{h}^0)$ is $D^2\times S^2$ so 

so $S^2\times S^2 - (4\text{-handle})$ is



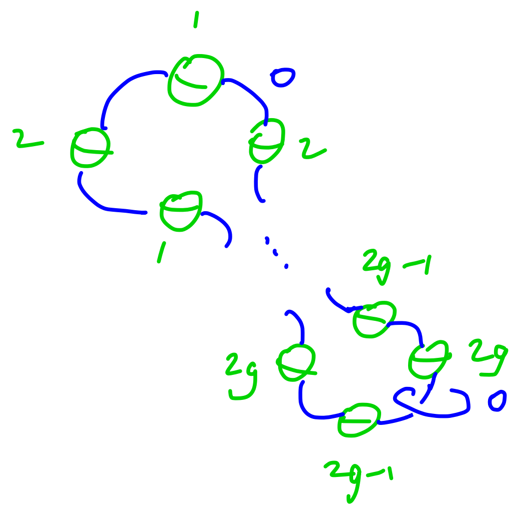
and $S^2\times S^2$ is  $\cup$ 4-handle

exercise: "see" 2 S^2's in $S^2\times S^2$ that
intersect in a point
ie. "see" $S^2\times\{p\}$ and $\{p\}\times S^2$
in the Kirby diagram

Hint: the link  bounds 2 disks in B^4 that intersect in a point

cap of disk using cores of 2-handles

8) exercise: show $\Sigma_g \times S^2$ is



$\cup$ 2g 3-handles

$\cup$ 4-handle

9) $\mathbb{C}P^2$

recall $\mathbb{C}P^2$ has coordinate charts

$$\phi_0 : \mathbb{C}^2 \to \{[z_0 : z_1 : z_2] \mid z_0 \neq 0\}$$

homogeneous coords

i.e. equiv. class of (z_0, z_1, z_2) in $\mathbb{C}^3 - \{0\}$ under action of $\mathbb{C} - \{0\}$

$$(z_1, z_2) \mapsto [1 : z_1 : z_2]$$

$$\phi_1 : \mathbb{C}^2 \to \{[z_0 : z_1 : z_2] \mid z_1 \neq 0\}$$

$$(w_1, w_2) \mapsto [w_1 : 1 : w_2]$$

ϕ_2 similar

unit disk

if we set $h^0 = D^2 \times D^2 \subset \mathbb{C}^2$

$h^2 = D^2 \times D^2 \subset \mathbb{C}^2$

$h^4 = D^2 \times D^2 \subset \mathbb{C}^2$

exercise:

1) $\phi_0(h^0), \phi_1(h^2), \phi_2(h^3)$ cover $\mathbb{C}P^2$

2) $\phi_0(h^0) \cap \phi_1(h^2) = \phi_0((\partial D^2) \times D^2)$

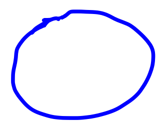
so $\phi_0(h^0) \cup \phi_1(h^2) \subset \mathbb{C}P^2$ is just $h^0 \cup h^2$

with $\partial D^2 \times D^2 \subset \partial h^2$ glued to $\partial D^2 \times D^2 \subset \partial h^0$

i.e. glued like 2-handle

as we saw above $\partial D^2 \times D^2 \subset \partial h^0 = S^3$

is just a nbhd of



so we see $h^0 \cup h^2$ will be a D^2-bundle over S^2!

(not surprising this will just be a

nbhd of $\underbrace{\mathbb{C}P^1}_{\cong S^2} \subset \mathbb{C}P^2$)

now h^0 and h^2 are glued by

$$\phi_0^{-1} \circ \phi_1 : \partial D^2 \times D^2 \to \partial D^2 \times D^2$$

$$(w_1, w_2) \mapsto (w_1^{-1}, w_1^{-1} w_2)$$

in notation above

$$(\phi, (r, \theta)) \mapsto (-\phi, (r, \theta - \phi))$$

exercise: this is E_1

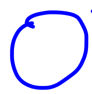
note: we know it is E_k some k but since we can glue h^4 to it to get $\mathbb{C}P^2$ we know its boundary must be S^3

$\therefore k = \pm 1$

you can directly argue for $k = +1$

or when we discuss "intersection forms" we will see that it must be $+1$ since "complex surfaces intersect positively"

so $\mathbb{C}P^2 = E_1 \cup$ 4-handle

i.e. $\mathbb{C}P^2 =$  ^{1} $\cup$ 4-handle

$\overline{\mathbb{C}P}^2 = \mathbb{C}P^2$ with opposite orientation and is given by

 $^{-1}$ $\cup$ 4-handle

10) Plumbing

let $\begin{array}{c} D^n \to E_i \\ \downarrow \\ M_i \end{array}$ be a D^n-bundle over an n-manifold M_i for $i = 0, 1$

choose a disk $D_i^n \overset{e_i}{\hookrightarrow} M_i$

we know $E_i|_{D_i^n} \cong D_i^n \times D^n$

let $f: E_0|_{D_0^n} \to E_1|_{D_1^n}$

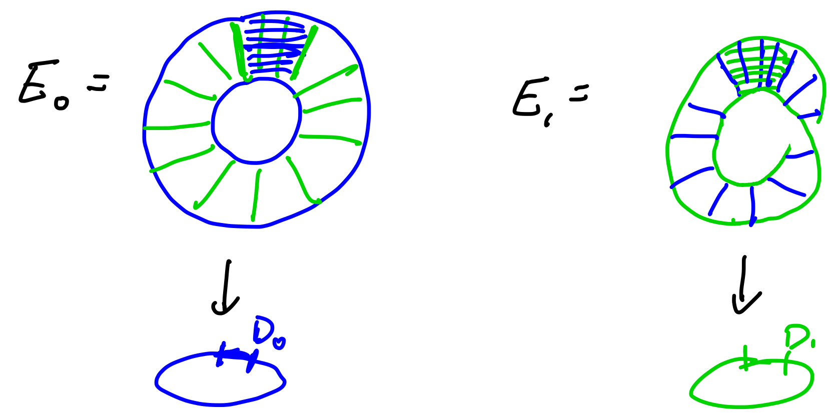
$(p,q) \mapsto (q,p)$

the <u>plumbing</u> of E_0 and E_1 is

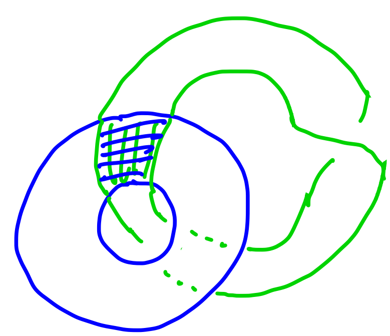
$$E_0 \cup_f E_1 = E_0 \amalg E_1 \big/ (p,q) \sim f(p,q)$$

(and round corners)

eg: <u>$n=1$</u>

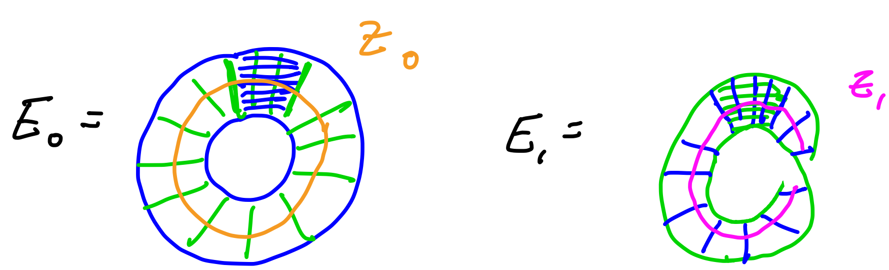


$P = E_0 \cup_f E_1$ is

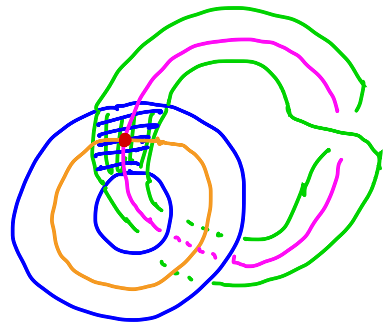


<u>exercise</u>: if M_0, M_1 connected the plumbing is independent of choices

<u>note</u>: in E_i we have the zero sections Z_i



in P we see Σ_0 and Σ_1 transversely intersect in one point



lemma 3:

if M_0^n, M_1^n are submanifolds of W^{2n} that intersect transversely in one point, then M_i^n has a disk bundle neighborhood E_i and $\exists$ a neighborhood of $M_0 \cup M_1$ is diffeomorphic to the plumbing of E_0 and E_1

Proof: exercise

in dimension 4, we know a disk bundle E_i over a surface of genus g_i is determined by an integer n_i

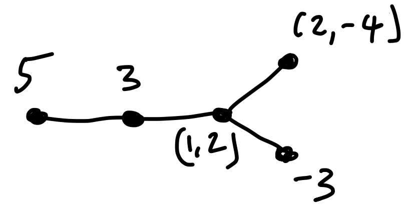
we denote the plumbing of E_0 and E_1 by the plumbing diagram

if $g_i = 0$, we usually leave it out of notation

is the plumbing of 2 disk bundles over S^2

of course we can plumb more than once

e.g.

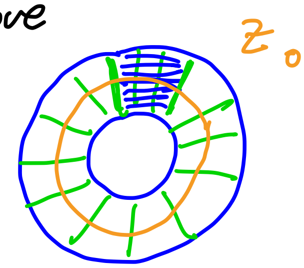


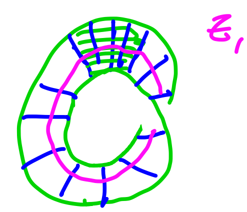
is the plumbing of 5 disk bundles and represents a nbhd of 5 surfaces in a 4-manifold intersecting according to the diagram

so what is a Kirby diagram for a plumbing?

lets start in dimension 2

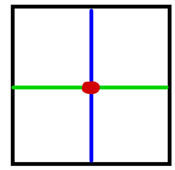
for the examples above

$E_0 =$ 

$E_1 =$ 

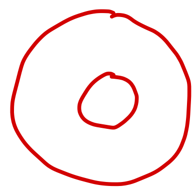
we have Kirby pictures

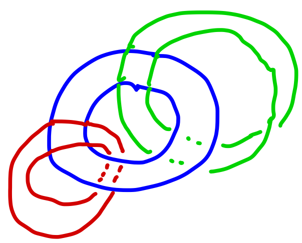
now a nbhd of the intersection (ie. plumbing region) is



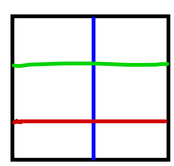
and 1-handle for green attached to ∂ green interval
1-handle for blue " " blue " "

so

suppose you plumbed  to blue



now nbhd of all intersections is

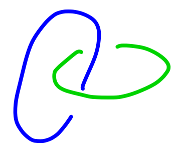


so diagram is

now for 4-manifolds

these are 2 D^2-bundles over S^2

the nbhd of the intersection point (i.e. plumbing region) is the 0-handle and ∂ of intersecting disks in $S^3 = \partial h^0$ is

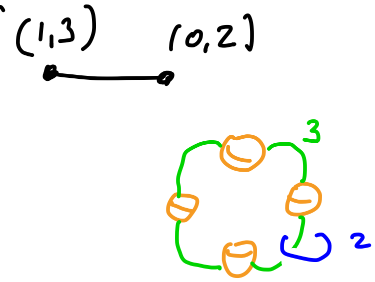


so the plumbed spheres are

exercise: "see" 2 S^2's in the picture and "see" them intersecting in one point

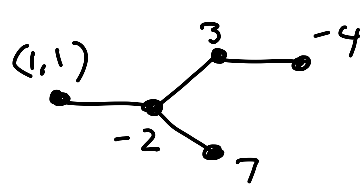
exercise: show plumbings of higher genus surfaces are similar

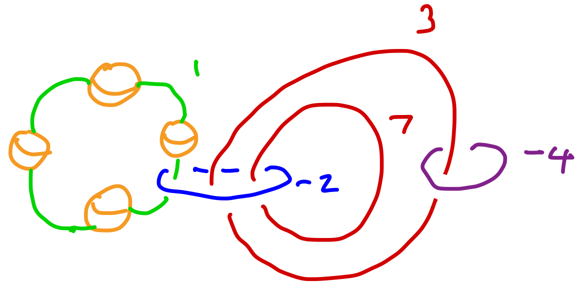
e.g.



and one can similarly consider plumbing many times

e.g.

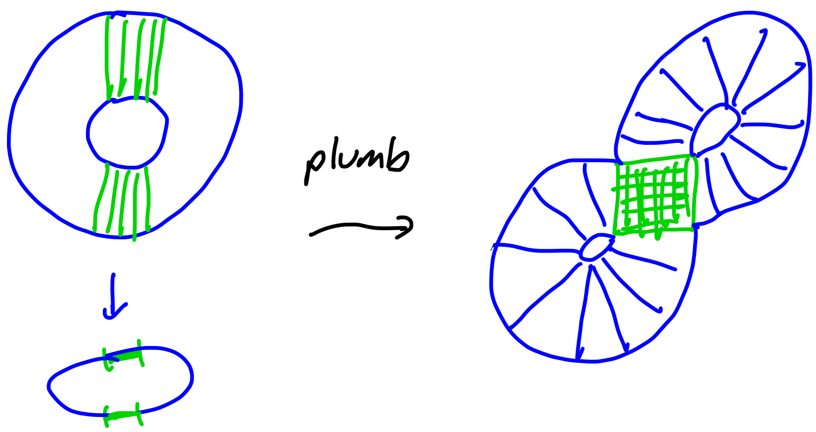




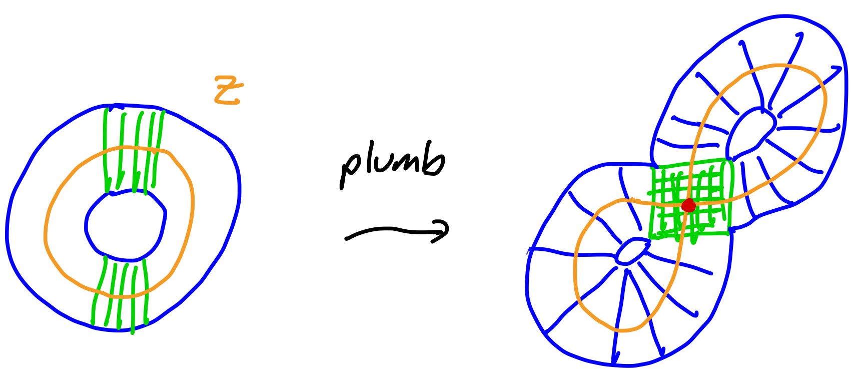
II) Self-Plumbing

given a disk bundle over a surface we can plumb it to itself

2D example



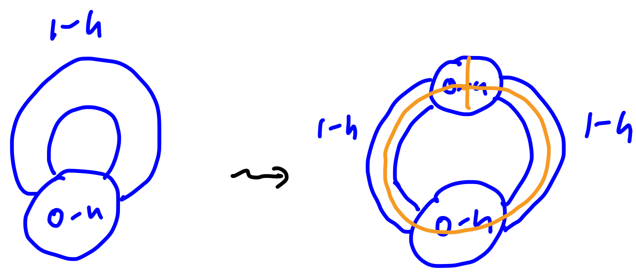
again, if we consider the zero section we see the plumbing as a nbhd of a circle with a transverse double point



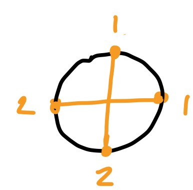
now for Kirby diagrams

recall for plumbings we identified the 0-handles of the two bundles so we saw the double point

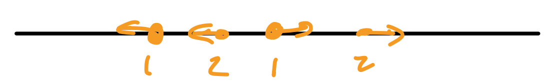
so we need to add a cancelling pair of 0,1-handles



identify 0-handles to get



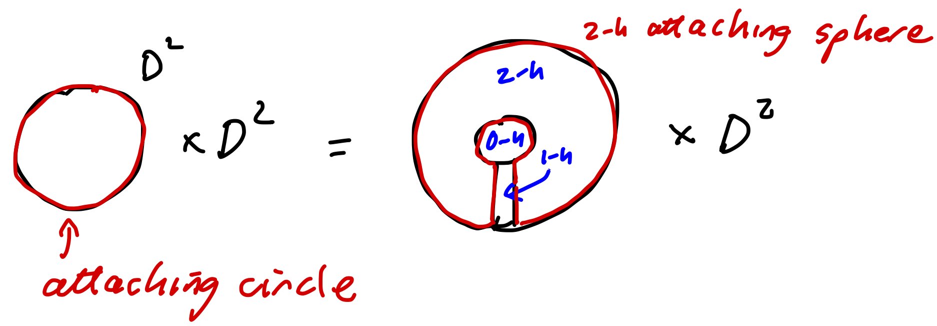
so diagram is



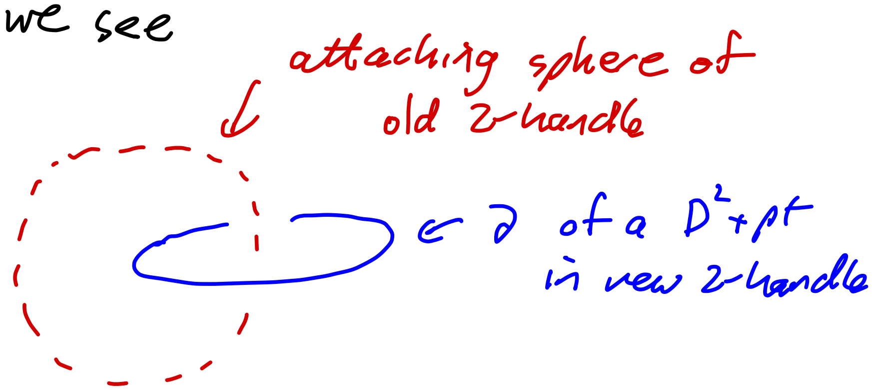
now for 4D:

to 0-h we need to add a cancelling 0/1-pair of handles and then the 2-handle

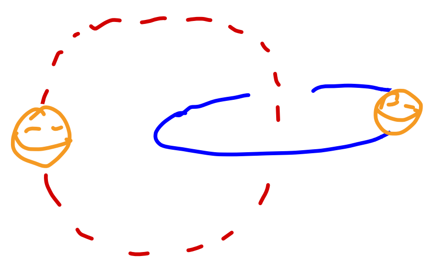
i.e. we think of the 2-handle as



now we identify new 0-handle with old one
so we see



we attach a 1-handle with one foot on red
and one on blue



now attach 2-handle so it runs over old
red and blue use framing coming
from bundle

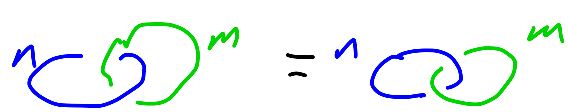
<u>note!</u> we have a clasp so could we
instead draw

Yes but these are different manifolds!

when you <u>self-plumb</u> there is a <u>sign</u>

i.e. if you orient the S^2 that is the zero section of the disk bundle then there is a sign for the double point! and if you change the orientation you <u>do not</u> change the sign!

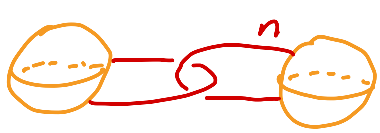
(<u>note</u>: for <u>non</u> self plumbings you can choose orientations so you get any sign for intersections of surfaces. In the diagram this is reflected in the fact that



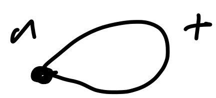
)

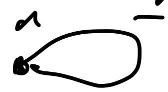
<u>exercise</u>: See that

corresponds to a sphere with a + double point



we denote the plumbing diagram

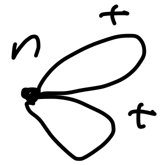
by 



respectively

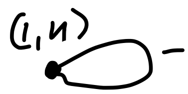
you can plumb multiple times and surfaces of higher genus

<u>exercise:</u>

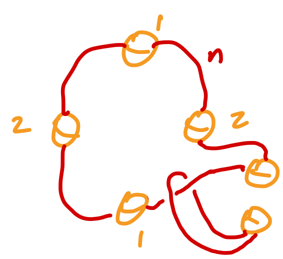
1) show 

is

generalize to any number of self-plumbings

2) Show 

is



and generalize to any genus and number of self-plumbings

<u>lemma 4:</u>

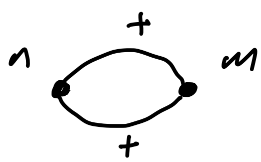
any surface Σ immersed in a 4-manifold X has a neighborhood that is a self-

plumbed disk bundle over Σ.

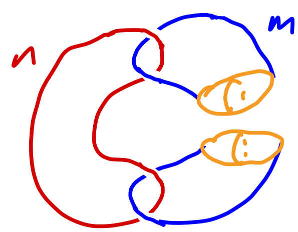
generalized self-plumbings:

any time you have a closed loop in a plumbing diagram

e.g.



exercise: Show a diagram for the above is



Show how to get general diagrams

eg try